

On a singular Fredholm–type integral equation arising in $\mathcal{N} = 2$ super–Yang–Mills theories

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In this work we study the Nekrasov–Shatashvili limit of the Nekrasov instanton partition function of Yang–Mills field theories with $\mathcal{N} = 2$ supersymmetry and gauge group $SU(N)$. The theories are coupled with fundamental matter. The equation that determines the density of eigenvalues at the leading order in the saddle-point approximation is exactly solved. The dominating contribution to the instanton free energy is computed. The requirement that this energy is finite imposes quantization conditions on the parameters of the theory that are in agreement with analogous conditions that have been derived in previous works. Using methods borrowed from the theory of matrix models, a field theoretical expression of the full instanton partition function is derived. It is checked that in the Nekrasov–Shatashvili (thermodynamic) limit the action of the field theory obtained in this way reproduces exactly the equation of motion used in the saddle-point calculations.

I. INTRODUCTION

In 1994 Seiberg and Witten proposed the solution for the low-energy effective $\mathcal{N} = 2$ supersymmetric gauge theories [1, 2]. This solution has led to considerable progress in understanding of the strong coupling dynamics of gauge theory. In recent days supersymmetric Yang–Mills theories with $\mathcal{N} = 2$ supersymmetry have become again a subject of intense studies mainly due to the discovery of the so-called AGT–W [3, 4] and Bethe/gauge correspondences [5–7]. A crucial ingredient of these conjectures is the multi-instanton Nekrasov partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ [8, 9] that takes into account the instanton sector of classes of four dimensional quiver gauge field theories with $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ gauge theory must be embedded in a Ω –background in order to apply the localization technique. With the help of this technique it is possible to express $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ in the form of a sum of contour integrals over k variables ϕ_I , $I = 1, \dots, k$, where the integer k ranges from zero to infinity. Here q denotes an effective scale, while ϵ_1 and ϵ_2 are the deformation parameters of the Ω –background. For a wide set of applications [5, 10–16] it is sufficient to evaluate $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ in the limit $\epsilon_2 \rightarrow 0$ while ϵ_1 is kept finite. This limit can be studied with the saddle-point method, which requires the solution of an infinite system of algebraic equations in the discrete variables ϕ_I mentioned above (cf. [10, 11]). This task is usually accomplished after applying some approximation, for instance by performing a series expansion in the parameter q and computing the coefficients of the expansion by a recursive algorithm [10]. Alternatively, by the

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introduction of the density function that describes the distribution of the ϕ_I 's, the saddle point equations may be cast in the form of a Fredholm equation on the continuous line. The density function becomes in fact continuous in the limit $\epsilon_2 \rightarrow 0$. Up to now, these Fredholm equations have been only marginally investigated, see for example [11]. The problem of finding their solutions will be the subject of the present work. To fix the ideas, we consider here the case of a $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with $SU(N)$ gauge group of local symmetry. The theory is coupled to matter in the fundamental representation and with a number N_f of flavors. The Fredholm equation that determines the configurations of the density functional at leading order in the saddle-point approximation are solved exactly. The expression of the instanton energy is evaluated. It is found that the consistency requirements that are necessary in order to keep the energy finite impose a quantization condition on the masses of the fundamental matter and on the vacuum expectation values of the adjoint scalar fields. This condition is in agreement with an analogous condition that has been postulated in previous works, see for example [13, 14]. Finally, we present a field theoretic expression of the Nekrasov instanton partition function in the limit $\epsilon_2 \rightarrow 0$ using techniques borrowed from the theory of matrix models [17].

The material of this paper is divided as follows. In the next Section the instanton partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ is studied in the thermodynamic limit, i.e. when $\epsilon_2 \rightarrow 0$ while ϵ_1 is kept finite. Following the arguments of [9] and statistical mechanics, it is shown that the dominant number \bar{k} of instantons is $\bar{k} = \frac{X}{\epsilon_2}$, where X is a constant to be determined. This constant is also directly related to the instanton corrections to the prepotential of the gauge theory and, via the Matone's identity [18], to the instanton corrections of the scalar field condensate. A field theoretical expression of the instanton partition function in the thermodynamic limit is provided. In Section 3 the saddle point equations are solved exactly. In order to determine the value of the X parameter mentioned above and the length of the interval in which the one-cut solutions are defined, a set of two implicit equations is provided. The instanton free energy is evaluated. The derivation of a field theory formulation of the instanton partition function is the subject of Section 4. Finally, our conclusions are drawn in Section 5.

II. CONTINUOUS DENSITY REPRESENTATION

In this work we consider the instanton partition function of a $\mathcal{N} = 2$ gauge field theory with gauge group $SU(N)$ and matter in the fundamental representation [9, 11]:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = 1 + \sum_{k=1}^{\infty} \frac{q^k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \int_{\mathbb{R}} \prod_{I=1}^k \frac{d\phi_I}{2\pi i} \prod_{I \neq J=1}^k D(\phi_I - \phi_J) \prod_{I=1}^k Q(\phi_I), \quad (2.1)$$

where:

$$D(z) = \frac{z(z + \epsilon_1 + \epsilon_2)}{(z + \epsilon_1)(z + \epsilon_2)}, \quad Q(z) = \frac{M(z)}{P(z + \epsilon_1 + \epsilon_2)P(z)}, \quad (2.2)$$

$$M(z) = \prod_{r=1}^{N_f} (z + m_r), \quad P(z) = \prod_{l=1}^N (z - a_l). \quad (2.3)$$

Here the a_l 's, $l = 1, \dots, N$, are the vacuum expectation values of the adjoint scalar field in the $SU(N)$ vector multiplet, while the m_r 's parametrize the masses of the fundamental matter. N_f can be identified with the number of flavors of the theory, though a more precise description of its meaning can be found in [8]. Let us note that the integrals over the real line \mathbb{R} in Eq. (2.1) require some form of regularization and should be intended as integrals over a closed contour. The details are explained in [8, 17].

In the following we will study the partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ in the “thermodynamic” limit $\epsilon_2 = 0$ that, according to [5], is related to the quantization of classical systems.

For very small values of ϵ_2 , Eq. (2.1) reduces to:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}} \prod_{I=1}^k \frac{d\phi_I}{2\pi i \epsilon_2} e^{\frac{1}{\epsilon_2} W_k(\{\phi_I\})}, \quad (2.4)$$

where

$$W_k(\{\phi_I\}) = \sum_{I>J=1}^k \epsilon_2^2 G(\phi_I - \phi_J) + \sum_{I=1}^k \epsilon_2 \log(q Q_0(\phi_I)) \quad (2.5)$$

with

$$G(\phi_I - \phi_J) = \frac{2\epsilon_1}{\epsilon_1^2 - (\phi_I - \phi_J)^2} \quad (2.6)$$

and

$$Q_0(z) = \frac{M(z)}{P(z + \epsilon_1)P(z)}. \quad (2.7)$$

Alternatively to Eq. (2.6), it is possible to write:

$$G(\phi_I - \phi_J) = \frac{\partial}{\partial \phi_I} \log \left(\frac{\phi_I - \phi_J + \epsilon_1}{\phi_I - \phi_J - \epsilon_1} \right) = -\frac{\partial}{\partial \phi_J} \log \left(\frac{\phi_I - \phi_J + \epsilon_1}{\phi_I - \phi_J - \epsilon_1} \right). \quad (2.8)$$

According to [9], in the thermodynamic limit $\epsilon_2 \rightarrow 0$ the instantonic partition function is dominated by the contributions for which

$$k = \bar{k} = \frac{X}{\epsilon_2} \quad (2.9)$$

i. e.:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim Z_{\bar{k}}(q, \epsilon_1, \epsilon_2) = \int_{\mathbb{R}} \prod_{I=1}^{\bar{k}} d\phi_I e^{\frac{1}{\epsilon_2} W_{\bar{k}}(\{\phi_I\})} \quad (2.10)$$

with $W_{\bar{k}}(\{\phi_I\})$ being defined in Eq. (2.5). In Eq. (2.9) X denotes a finite proportionality constant. Eq. (2.10) is in agreement with the predictions of statistical mechanics. Indeed, $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ may be regarded as the grand canonical partition function of a system with potential energy $W_k(\{\phi_I\})$, with k playing the role of the number of particles in the system. It is well known that in the thermodynamic limit the grand canonical partition function is dominated by the contribution corresponding

to the most probable number of particles \bar{k} . From now on, we will consider the instantonic partition function in the thermodynamic limit (2.10).

Following [19, 20], we perform now the continuous limit of the instantonic partition function by introducing the new field variable $\phi(I\epsilon_2)$ defined by the relations: $\phi_I = \phi(I\epsilon_2)$, $I = 1, \dots, \bar{k}$. Of course, when $I = \bar{k}$ we have that $I\epsilon_2 = X$. We note also that the quantity $x_I = I\epsilon_2$ may be considered as a discrete coordinate in the interval $[0, X]$. It is easy to realize that in terms of the field $\phi(x_I)$ the instantonic partition function becomes:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim \int_{\mathbb{R}} \prod_{I=1}^{\bar{k}} \frac{d\phi(x_I)}{2\pi i} \exp \left[\frac{1}{\epsilon_2} W_{\bar{k}}(\{\phi(x_I)\}) \right] \quad (2.11)$$

with

$$W_{\bar{k}}(\{\phi(x_I)\}) = \sum_{I < J=1}^{\bar{k}} \epsilon_2^2 G(\phi(x_I) - \phi(x_J)) + \sum_{I=1}^{\bar{k}} \epsilon_2 \log(qQ_0(\phi(x_I))) \quad (2.12)$$

If ϵ_2 is vanishingly small, x_I may be replaced by the continuous variable $x \in [0, X]$ and the expression of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ may formally be written as follows:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = \int \mathcal{D}\phi(x) \exp \left[\frac{1}{\epsilon_2} W_{\bar{k}}(\phi(x)) \right], \quad (2.13)$$

where

$$\mathcal{D}\phi(x) = \prod_{I=1}^{\bar{k}} \frac{d\phi(x_I)}{2\pi i \epsilon_2} \quad (2.14)$$

and

$$W_{\bar{k}}(\phi(x)) = \int_0^X dx \int_0^x dy G(\phi(x) - \phi(y)) + \int_0^X dx \log(qQ_0(\phi(x))). \quad (2.15)$$

In the integrals over x and y of Eq. (2.15) the principal value prescription is implicitly understood in order to cure the singularity of $G(z)$ at the points $z = \pm\epsilon_1$.

III. SADDLE POINT EVALUATION

For small values of ϵ_2 , the summation over all configurations $\phi(x)$ in Eq. (2.13) will be dominated by those configurations $\phi_{\text{cl}}(x)$ that correspond to the absolute maximum of the \bar{k} -instantonic energy $W_{\bar{k}}(\phi(x))$, i. e.:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim \exp \left[\frac{1}{\epsilon_2} W_{\bar{k}}(\phi_{\text{cl}}(x)) \right]. \quad (3.16)$$

In order to find the extremal points of the instantonic energy, we introduce the density $\rho(\phi)$ defined by the relation:

$$d\phi\rho(\phi) = dx. \quad (3.17)$$

Eq. (3.17) implies also the condition:

$$\int_{-\infty}^{+\infty} d\phi \rho(\phi) = X. \quad (3.18)$$

With the help of (3.17), the functional $W_{\bar{k}}(\phi(x))$ of Eq. (2.15) can be expressed as a functional of $\rho(\phi)$:

$$e^{\frac{1}{\epsilon_2} W_{\bar{k}}(\phi(x))} = \exp \left[\frac{1}{\epsilon_2} \left(\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) \rho(\phi') G(\phi - \phi') + \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(q Q_0(\phi)) \right) \right], \quad (3.19)$$

where

$$\int_{-\infty}^{+\infty} f(\phi) d\phi$$

denotes the principal value of the integral of a generic function $f(\phi)$. Supposing that $f(\phi)$ has singularities at infinity and at some points a_σ , $\sigma = 1, \dots, s$, the principal value is defined as follows:

$$\int_{-\infty}^{+\infty} f(\phi) d\phi = \lim_{R \rightarrow +\infty} \lim_{\mu_1, \dots, \mu_s \rightarrow 0} \left[\int_{-R}^{a_1 - \mu_1} f(\phi) d\phi + \sum_{\sigma=2}^{s-1} \int_{a_{\sigma-1} + \mu_{\sigma-1}}^{a_\sigma - \mu_\sigma} f(\phi) d\phi + \int_{a_s + \mu_s}^R f(\phi) d\phi \right] \quad (3.20)$$

Moreover, in order to obtain Eq. (3.19) from Eq. (2.15) we have used the identity:

$$\int_0^X dx \int_0^x dy G(\phi(x) - \phi(y)) = \frac{1}{2} \int_0^X dx \int_0^X dy G(\phi(x) - \phi(y)). \quad (3.21)$$

The stationary points of the exponent present in the right hand side of Eq. (3.19) are given by the solutions of the saddle point equation:

$$\int_{-\infty}^{+\infty} d\phi' \rho(\phi') G(\phi - \phi') + \log(q Q_0(\phi)) = 0. \quad (3.22)$$

Once a solution $\rho_{\text{cl}}(\phi)$ of the above equation is found, the corresponding configurations $\phi_{\text{cl}}(x)$ appearing in Eq. (3.16) that maximize the energy $W_{\bar{k}}(\phi_{\text{cl}}(x))$ may be recovered using Eq. (3.17). To solve Eq. (3.22) it is possible to exploit the superposition principle and split $\rho(\phi)$ as follows:

$$\rho(\phi) = \rho_q(\phi) + \rho_{Q_0}(\phi), \quad (3.23)$$

where $\rho_q(\phi)$ and $\rho_{Q_0}(\phi)$ are respectively the solutions of the equations:

$$\int_{-\infty}^{+\infty} d\phi' \rho_q(\phi') G(\phi - \phi') + \log(q) = 0 \quad (3.24)$$

and

$$\oint_{-\infty}^{+\infty} d\phi' \rho_{Q_0}(\phi') G(\phi - \phi') + \log(Q_0(\phi)) = 0. \quad (3.25)$$

Let us consider Eq. (3.24) first. A similar equation, namely a Fredholm integral equation of the first kind, has been discussed in [21], Section 12.2. Naively, one could try the solution $\rho_q(\phi) = A\phi + B$, A, B being constants. With this ansatz, however, Eq. (3.24) is satisfied only if $A = B = 0$, which implies the undesirable condition $\log(q) = 0$. Thus, following [21], we suppose that the support of $\rho_q(\phi)$ is limited to the finite interval $[-L, L]$ on the ϕ -axis. In this way, we introduce a new constant L , whose value will be fixed later. After some calculations, whose details are given in the Appendix, one finds:

$$\rho_q(\phi) = -\frac{\log(q)}{2\epsilon_1\pi} \sqrt{L^2 - \phi^2} [\theta(L - \phi) - \theta(-L - \phi)], \quad (3.26)$$

where $\theta(x)$ denotes the Heaviside theta function.

At this point we are ready to solve Eq. (3.25). To this purpose, it is convenient to pass to the Fourier representation. Denoting with $\tilde{\rho}_{Q_0}(\omega)$, $\tilde{G}(\omega)$ and $\lambda(\omega)$ respectively the Fourier transforms of $\rho_q(\phi)$, $G(\phi - \phi')$ and $\log(Q_0(\phi))$, we obtain from Eq. (3.25) the following identity:

$$\sqrt{2\pi} \tilde{\rho}_{Q_0}(\omega) \tilde{G}(\omega) = -\lambda(\omega). \quad (3.27)$$

Before we proceed, a remark concerning the computation of the Fourier transform of $\log(Q_0(\phi))$ is in order. To this purpose, we have in fact to evaluate the Fourier transform of logarithmic functions [22, 23] which are not uniquely defined [24]. To avoid this problem, we assume here the condition:

$$N_f = 2N. \quad (3.28)$$

Under this condition, the function $\log(Q_0(\phi))$ has no singularities at infinity and its Fourier transform can be determined without ambiguities. After some calculations, which are presented in the Appendix, we arrive at the following results:

$$\tilde{G}(\omega) = +\sqrt{2\pi} \operatorname{sgn}\omega \sin(\epsilon_1\omega) \quad (3.29)$$

and

$$\lambda(\omega) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\omega} \operatorname{sgn}\omega \left(\sum_{l=1}^N e^{i\omega a_l} + \sum_{l=1}^N e^{i\omega(a_l - \epsilon_1)} - \sum_{r=1}^{N_f} e^{i\omega m_r} \right), \quad (3.30)$$

where the sign function $\operatorname{sgn}\omega$ is defined as follows:

$$\operatorname{sgn}\omega = \begin{cases} 1 & \text{if } \omega > 0 \\ 0 & \text{if } \omega = 0 \\ -1 & \text{if } \omega < 0 \end{cases} \quad (3.31)$$

After substituting Eqs. (3.29) and (3.30) in Eq. (3.27), it is easy to derive $\tilde{\rho}_{Q_0}(\omega)$ for $\omega \neq 0$:

$$\tilde{\rho}_{Q_0}(\omega) = \frac{1}{2\omega \sin \omega \epsilon_1} \frac{1}{\sqrt{2\pi}} \left(\sum_{r=1}^{N_f} e^{-i\omega m_r} - \sum_{l=1}^N e^{i\omega a_l} - \sum_{l=1}^N e^{i\omega(a_l - \epsilon_1)} \right). \quad (3.32)$$

When $\omega = 0$, Eq. (3.27) becomes undefined, because $\tilde{G}(\omega)$ and $\lambda(\omega)$ are both proportional to $\text{sgn}\omega$ which is zero when $\omega = 0$. Due to the normalization condition (3.18), $\tilde{\rho}_{Q_0}(0)$ should be finite. As a matter of fact, Eq. (3.18) implies:

$$-\frac{L^2 \log(q)}{4\epsilon_1} + \sqrt{2\pi} \tilde{\rho}_{Q_0}(0) = X. \quad (3.33)$$

In order to derive the above equation, we have used the fact that

$$\int_{-\infty}^{+\infty} \rho_{Q_0}(\phi) d\phi = \sqrt{2\pi} \tilde{\rho}_{Q_0}(0) \quad (3.34)$$

and

$$\int_{-\infty}^{+\infty} \rho_q(\phi) d\phi = -\frac{L^2 \log(q)}{4\epsilon_1}. \quad (3.35)$$

On the other side, the double singularity of $\tilde{\rho}_{Q_0}(\omega)$ in $\omega = 0$, which is present in the expression of $\tilde{\rho}_{Q_0}(\omega)$ given in Eq. (3.32), is not treatable even with the prescription of the principal value. As a consequence, it turns out that $\lim_{\omega \rightarrow 0} |\tilde{\rho}_{Q_0}(\omega)| = +\infty$ for arbitrary values of the parameters a_l and m_r . To make $\tilde{\rho}_{Q_0}(\omega)$ a continuous function in $\omega = 0$ and Eqs. (3.32) and (3.33) consistent with each other, it is necessary the introduction of the following additional condition on the parameters a_l 's and m_r 's:

$$\sum_{r=1}^{N_f} m_r + 2 \sum_{l=1}^N a_l = N\epsilon_1. \quad (3.36)$$

Under this condition we find from Eqs. (3.32) after a simple expansion near the point $\omega = 0$ that:

$$\tilde{\rho}_{Q_0}(0) = \frac{1}{4\epsilon_1} \left[\sum_{l=1}^N (a_l^2 + (a_l + \epsilon_1)^2) - \sum_{r=1}^{N_f} m_r^2 \right]. \quad (3.37)$$

It is now possible to derive a new relation which, together with Eq. (3.33), will be able to determine the still unknown parameters X and L . Indeed, with the help of equations (3.24) and (3.25), the following identity can be proved:

$$\tilde{\rho}_{Q_0}(0) = -\frac{1}{2\pi\epsilon_1} \int_{-L}^{+L} d\phi' \sqrt{L^2 - \phi'^2} \log(Q_0(\phi')), \quad (3.38)$$

i. e.:

$$\int_{-L}^L d\phi' \sqrt{L^2 - \phi'^2} \log(Q_0(\phi')) = -\frac{\pi}{2} \left(\sum_{l=1}^N (a_l^2 + (a_l - \epsilon_1)^2) - \sum_{r=1}^{N_f} m_r^2 \right). \quad (3.39)$$

Eq. (3.39) provides an implicit equation which allows to derive the value of L as a function of the parameters of the $\mathcal{N} = 2$ gauge field theory. Once the value of L is known, it is possible to determine also the value of X thanks to the normalization condition (3.33). As a final remark concerning the expression of $\tilde{\rho}_{Q_0}(\omega)$, we note in Eq. (3.32) the presence of additional singularities at the points

$$\omega = \frac{\pi\sigma}{\epsilon_1} \quad \sigma = \pm 1, \pm 2, \dots \quad (3.40)$$

They appear due to the presence of the function $\sin(\omega\epsilon_1)$ in the denominator of Eq. (3.32). These singularities do not pose problems and, whenever it is necessary to perform an integration involving $\tilde{\rho}_{Q_0}(\omega)$, they can be cured with the help of the principal value prescription. There is an infinite number of these singularities, but their contributions after taking the principal value prescription amount to a series which is convergent.

Next, we are going to compute the energy:

$$W = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi') + \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(q Q_0(\phi)) \quad (3.41)$$

appearing in the right hand side of Eq. (3.19). Since we restrict ourselves to the stationary configurations $\rho(\phi)$ satisfying the saddle point equations (3.22), it is possible to rewrite the above energy in the more compact form:

$$W = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \rho(\phi) (\log q + \log(Q_0(\phi))). \quad (3.42)$$

Taking into account also the split of the density $\rho(\phi)$ of Eq. (3.23) and the fact that at the stationary point the following identity holds:

$$\int_{-\infty}^{+\infty} d\phi \rho_q(\phi) \log(Q_0(\phi)) = \log q \int_{-\infty}^{+\infty} d\phi \rho_{Q_0}(\phi) \quad (3.43)$$

we arrive at the following convenient expression of W :

$$W = W_{qq} + W_{qQ_0} + W_{Q_0Q_0}, \quad (3.44)$$

where

$$W_{qq} = \frac{\log q}{2} \int_{-\infty}^{+\infty} d\phi \rho_q(\phi), \quad (3.45)$$

$$W_{qQ_0} = \log q \int_{-\infty}^{+\infty} d\phi \rho_{Q_0}(\phi), \quad (3.46)$$

$$W_{Q_0Q_0} = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \rho_{Q_0}(\phi) \log(Q_0(\phi)). \quad (3.47)$$

A straightforward calculation shows that:

$$W_{qq} = -\frac{L^2(\log q)^2}{8\epsilon_1}. \quad (3.48)$$

W_{qQ_0} may be computed by remembering that

$$\oint_{-\infty}^{+\infty} d\phi \rho_{Q_0}(\phi) = \sqrt{2\pi} \tilde{\rho}_{Q_0}(0)$$

due to Eq. (3.34). The expression of $\tilde{\rho}_{Q_0}(0)$ is already known from Eq. (3.37), so that it is possible to write:

$$W_{qQ_0} = \frac{\sqrt{2\pi} \log q}{4\epsilon_1} \left[\sum_{l=1}^N (a_l^2 + (a_l - \epsilon_1)^2) - \sum_{r=1}^{N_f} m_r^2 \right] \quad (3.49)$$

W_{qq} and W_{qQ_0} are the most important contributions to the energy from the physical point of view because they contain the parameter q . Finally, to derive the explicit expression of $W_{Q_0Q_0}$ it is necessary to compute an integral in $d\phi$ that is too complicated to be evaluated in closed form.

IV. FIELD THEORETICAL FORMULATION

We start from Eq. (2.1) rewritten in a more appropriate way for our purposes:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = \sum_{k=0}^{+\infty} \frac{q^k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \int_{\mathbb{R}} \prod_{I=1}^k \frac{d\phi_I}{2\pi i} e^{H_k}, \quad (4.50)$$

where

$$H_k = \sum_{I \neq J=1}^k \log(D(\phi_I - \phi_J)) + \sum_{I=1}^k \log(Q(\phi_I)). \quad (4.51)$$

It is now possible to rewrite $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ as follows [17]:

$$\begin{aligned} Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) = & \sum_{k=0}^{+\infty} \frac{q^k}{k!} \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \int_{\mathbb{R}} \prod_{I=1}^k \frac{d\phi_I}{2\pi i} \int \mathcal{D}\rho(\phi) \delta \left(\rho(\phi) - \sum_{I=1}^k \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_I) \right) \\ & \times \exp \left[\oint_{-\infty}^{+\infty} d\phi d\phi' \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') \right. \\ & \left. + \oint_{-\infty}^{+\infty} d\phi \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right) \rho(\phi) \log(Q(\phi)) \right] \end{aligned} \quad (4.52)$$

In the above equation the principal value prescription of the integrals over the variable ϕ is necessary in order to avoid the singularities of $\log(D(\phi - \phi'))$ when $\phi = \phi'$. This is the analog in the continuous

case of the condition $I \neq J$ in the double discrete sum of Eq. (4.51). After introducing the Fourier representation of the Dirac delta function appearing in the above equation:

$$\delta\left(\rho(\phi) - \sum_{I=1}^k \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_I)\right) = \int \mathcal{D}\lambda \exp \left[i \int_{-\infty}^{+\infty} d\phi \lambda(\phi) \left(\rho(\phi) - \sum_{I=1}^k \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_\sigma) \right) \right] \quad (4.53)$$

we obtain:

$$\begin{aligned} Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) &= \sum_{k=0}^{+\infty} \frac{q^k}{k!} \int \mathcal{D}\rho \mathcal{D}\lambda \left(\int_{\mathbb{R}} \frac{d\phi}{2\pi i} \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} e^{-i \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \lambda(\phi)} \right)^k \\ &\times \exp \left[\int_{-\infty}^{+\infty} d\phi d\phi' \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') \right. \\ &\left. + \int_{-\infty}^{+\infty} d\phi \left(i\lambda(\phi) \rho(\phi) + \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \rho(\phi) \log(Q(\phi)) \right) \right]. \quad (4.54) \end{aligned}$$

Let us note that the contour integration over \mathbb{R} is now restricted to the integral of the quantity $e^{-i\lambda(\phi)}$. As mentioned in [17], the choice of the contour is not important when performing integrals over densities like $\rho(\phi)$ or $\lambda(\phi)$, so that it is possible to replace in Eq. (4.54) the contour integration with the integration over the whole real line, i. e.:

$$\int_{\mathbb{R}} \rightarrow \int_{-\infty}^{+\infty}$$

The summation over the k indexes in Eq. (4.54) can be easily performed and gives as a result:

$$\begin{aligned} Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) &= \int \mathcal{D}\rho \mathcal{D}\lambda \exp \left\{ \int_{-\infty}^{+\infty} d\phi d\phi' \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') \right. \\ &\left. + \int_{-\infty}^{+\infty} d\phi \left[i\rho(\phi) \lambda(\phi) + \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right) \left(\rho(\phi) \log(Q(\phi)) + q e^{-i \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \lambda(\phi)} \right) \right] \right\}. \quad (4.55) \end{aligned}$$

For future purposes, it will be convenient to perform in Eq. (4.55) the following shift of the auxiliary field $\lambda(\phi)$:

$$\lambda(\phi) = \lambda'(\phi) - i \frac{(\epsilon_1 + \epsilon_2)}{\epsilon_1 \epsilon_2} \log(q). \quad (4.56)$$

After the above shift, the instanton partition function $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ takes the form:

$$\begin{aligned} Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) &= \int \mathcal{D}\rho \mathcal{D}\lambda' \exp \left\{ \int_{-\infty}^{+\infty} d\phi d\phi' \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^2 \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') \right. \\ &\left. + \int_{-\infty}^{+\infty} d\phi \left[i\rho(\phi) \lambda'(\phi) + \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right) \left(\rho(\phi) \log(qQ(\phi)) + e^{-i \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \lambda'(\phi)} \right) \right] \right\}. \quad (4.57) \end{aligned}$$

This is the desired field theoretical expression of the instanton partition function. Let us note that we could have arrived to Eq. (4.57) putting in Eq. (4.52):

$$q^k = e^{k \log(q)} = e^{\sum_{I=1}^k \log(q)} = e^{\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \sum_{I=1}^k \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \log(q)}. \quad (4.58)$$

Using the Dirac delta function imposing the condition $\rho(\phi) = \sum_{I=1}^k \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2} \delta(\phi - \phi_I)$ we may finally write:

$$q^k = \exp \left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(q) \right). \quad (4.59)$$

Doing in this way it is possible to obtain Eq. (4.57) from (4.52) directly without the shift (4.56).

In order to pass to the limit $\epsilon_2 \rightarrow 0$, the following two formulas will be useful:

$$\begin{aligned} & \int_{-\infty}^{+\infty} d\phi d\phi' \rho(\phi) \log(D(\phi - \phi')) \rho(\phi') = \\ & \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{\phi} d\phi' \rho(\phi) [\log(D(\phi - \phi')) - \log(D(\phi' - \phi))] \rho(\phi') \end{aligned} \quad (4.60)$$

and

$$\lim_{\epsilon_2 \rightarrow 0} \left[\frac{\log(D(\phi - \phi')) + \log(D(\phi' - \phi))}{\epsilon_2} \right] = \frac{2\epsilon_1}{\epsilon_1^2 - (\phi - \phi')^2}. \quad (4.61)$$

With the help of the above equations it is easy to prove that, when $\epsilon_2 \sim 0$, the expression of $Z_{\text{inst}}(q, \epsilon_1, \epsilon_2)$ given in Eq. (4.57) may be approximated as shown below:

$$Z_{\text{inst}}(q, \epsilon_1, \epsilon_2) \sim \int \mathcal{D}\rho(\phi) \exp \left\{ \frac{1}{\epsilon_2} \left[\frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi') + \int_{-\infty}^{+\infty} d\phi \rho(\phi) \log(q Q_0(\phi)) \right] \right\}. \quad (4.62)$$

Let us notice that the dependence on the auxiliary field $\lambda(\phi)$ disappears in the limit $\epsilon_2 \rightarrow 0$. The above equation is in agreement with the classical approximation given in (3.19) and with the previous results, see for example Ref. [11]. In order to derive Eq. (4.62) we have used the fact that:

$$\int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{\phi} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi') = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \int_{-\infty}^{+\infty} d\phi' \rho(\phi) G(\phi - \phi') \rho(\phi'). \quad (4.63)$$

V. CONCLUSIONS

In this work the exact solution of the saddle point equations extremizing the density of eigenvalues $\rho(\phi)$ has been computed. The result is given in Eqs. (3.26) and (3.32). Let us note that the solution depend on two parameters, namely the half length L of the interval in which the one-cut

solution $\rho_q(\phi)$ is defined and the parameter X given in Eq. (2.9). The latter parameter plays an important role in the Seiberg–Witten theory, because the prepotential \mathcal{F} of the supersymmetric gauge theory satisfies the relation:

$$q \frac{\partial \mathcal{F}}{\partial q} = -X\epsilon_1. \quad (5.64)$$

The quantity $q \frac{\partial \mathcal{F}}{\partial q}$ is also related to the well known Matone’s relation [18]. Both parameters X and L may be extracted from the implicit relations provided by Eqs. (3.33), (3.37) and (3.39). The physically relevant part of the instanton energy in the thermodynamic limit has been computed. The result of these calculations is reported in Eqs. (3.48) and (3.49). Let us note that the requirement that the instanton free energy is finite leads to the quantization condition of Eq. (3.36). Finally, a field theoretical formulation of the full instanton partition function has been derived in Section 4, see Eq. (4.55). In the semiclassical limit, valid when ϵ_2 is very small, the leading order contribution in the partition function is provided in Eq. (4.62), in agreement with the arguments of [9] and statistical mechanics. In the future it is planned to study the path integral appearing in Eq. (4.55) using the method of Ref. [25] that allows to perform analytic calculations in the case of theories with nonpolynomial and complex potentials.

VI. ACKNOWLEDGMENTS

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Appendix A: Derivation of Eqs. (3.26), (3.29) and (3.30)

1. Derivation of the solution $\rho_q(\phi)$ of Eq. (3.24)

Since it is not possible to define $\rho_q(\phi)$ over the whole real line, we search for a so-called one-cut solution of Eq. (3.24) in the domain $\phi \in [-L, L]$. Accordingly, we choose $\rho_q(\phi)$ as follows:

$$\rho_q(\phi) = \rho'_q(\phi)[\theta(L - \phi) - \theta(-L - \phi)], \quad (A.1)$$

where $\theta(x)$ denotes the Heaviside function. After substituting the above expression of $\rho_q(\phi)$ in Eq. (3.24), we obtain a new equation for $\rho'_q(\phi)$:

$$\int_{-L}^L d\phi' \rho'_q(\phi') \left(\frac{1}{\phi - \phi' - \epsilon_1} - \frac{1}{\phi - \phi' + \epsilon_1} \right) = \log(q). \quad (A.2)$$

Performing the change of variable $y = \frac{\phi'}{L}$ and putting:

$$\rho''_q(y) = \rho'_q(Ly) = \rho'_q(\phi) \quad (A.3)$$

we obtain from Eq. (A.2):

$$\oint_{-1}^{+1} dy \rho_q''(y) \left(\frac{1}{\frac{\phi - \epsilon_1}{L} - y} - \frac{1}{\frac{\phi + \epsilon_1}{L} - y} \right) = \log(q). \quad (\text{A.4})$$

The above equation may be rewritten as follows:

$$\rho_q''^H \left(\frac{\phi - \epsilon_1}{L} \right) - \rho_q''^H \left(\frac{\phi + \epsilon_1}{L} \right) = \log(q), \quad (\text{A.5})$$

where $\rho_q''^H(z)$ denotes the finite Hilbert transform in the interval $[-1, +1]$ of the function $\rho_q''(y)$. To solve Eq. (A.4) or equivalently (A.5) it is useful to recall the following integral formula ¹:

$$\oint_{-1}^{+1} \frac{\sqrt{1-y^2}}{a-y} dy = a\pi. \quad (\text{A.6})$$

Looking at the above equation it is easy to realize that Eq. (A.4) is satisfied by:

$$\rho_q''(y) = -\frac{L \log(q)}{2\epsilon_1 \pi} \sqrt{1-y^2}. \quad (\text{A.7})$$

It is now possible to go back to the solution $\rho_q(\phi)$ of the original equation (3.24) with the help of Eqs. (A.1) and (A.3). The final result is:

$$\rho_q(\phi) = -\frac{\log(q)}{2\epsilon_1 \pi} \sqrt{L^2 - \phi^2} [\theta(L - \phi) - \theta(-L - \phi)]. \quad (\text{A.8})$$

2. Derivation of the Fourier transform $\tilde{G}(\omega)$ of Eq. (3.29)

The Fourier transform $\tilde{G}(\omega)$ of the function $G(\phi)$ defined in Eq. (2.6) is given by:

$$\tilde{G}(\omega) = \int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} e^{i\omega\phi} \left(\frac{1}{\phi + \epsilon_1} - \frac{1}{\phi - \epsilon_1} \right). \quad (\text{A.9})$$

In order to perform the integral in the right hand side of the above equation the following formula will be useful:

$$\int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} e^{i\omega\phi} \frac{1}{\phi - a} = e^{ia\omega} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} i \text{sgn} \omega. \quad (\text{A.10})$$

With the help of Eq. (A.10) it is easy to show that Eq. (3.29) holds.

¹ This integral appears in a technique developed by Tricomi to solve certain integral equations, see [21], chapter 11.4

3. Derivation of the Fourier transform $\lambda(\omega)$ of Eq. (3.30)

To compute the Fourier transform $\lambda(\omega)$ of the function $\log(Q_0(\phi))$, we note that under the condition (3.28) $\log(Q_0(\phi))$ may be expressed as follows:

$$\log(Q_0(\phi)) = \sum_{\sigma=1}^{N_f/2} \log\left(\frac{\phi - A_\sigma}{\phi - B_\sigma}\right) + \sum_{\sigma=1}^{N_f/2} \log\left(\frac{\phi - A_{N_f/2+\sigma}}{\phi - B_\sigma + \epsilon_1}\right), \quad (\text{A.11})$$

where

$$A_r = -m_r \quad \text{for} \quad r = 1, \dots, N_f, \quad (\text{A.12})$$

$$B_l = a_l \quad \text{for} \quad l = 1, \dots, N. \quad (\text{A.13})$$

It is clear from Eq. (A.11) that the building blocks of $\log(Q_0(\phi))$ are differences of pairs of logarithmic functions, whose arguments are linear in ϕ . As a consequence, the essential ingredient in the computation of the Fourier transform of $\log(Q_0(\phi))$ is the Fourier transform of such pairs. For generic constants A and B , this is given by:

$$\int_{-\infty}^{+\infty} \frac{d\phi}{\sqrt{2\pi}} e^{i\omega\phi} \log\left(\frac{\phi - A}{\phi - B}\right) = \left(\frac{\pi}{2}\right) \frac{\text{sgn}\omega}{\omega} (e^{i\omega B} - e^{i\omega A}). \quad (\text{A.14})$$

Applying Eq. (A.14) it is possible to verify that the expression of $\lambda(\omega)$ is exactly that given in Eq. (3.30).

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